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Note

A Technique for Solving the Non-Linear Partial Differential Equations of Tokamak Transport*

Because of the large parallel heat conductivity and long time scale associated with tokamaks, the equations describing their evolution can be expressed as a set of one-dimensional equations depending parametrically upon the results of a single two-dimensional equation [1]. The latter (the Grad–Shafranov equation) can be solved by standard techniques. The non-linear one-dimensional equations which remain are solved by the *G2M* code using the new technique described below.

The equations may be written in a number of essentially equivalent coordinate systems, all involving functions constant on a flux surface. In particular, using the poloidal flux as the independent coordinate gives as derived in Ref. [1]:

$$\frac{\partial n(\partial V/\partial\psi)}{\partial t}\bigg|_\psi + \frac{\partial}{\partial\psi}\langle \Gamma_R \cdot \nabla V\rangle = S_n \frac{\partial V}{\partial\psi} \tag{1}$$

$$\frac{3}{2}\frac{\partial P_e(\partial V/\partial\psi)}{\partial t}\bigg|_\psi + \frac{\partial}{\partial\psi}\left[P_e \frac{\partial V}{\partial t}\bigg|_\psi + \left\langle\left(Q_e + \frac{5}{2}\Gamma_R T_e\right)\cdot\nabla V\right\rangle\right] = S_e \frac{\partial V}{\partial\psi} \tag{2}$$

$$\frac{3}{2}\frac{\partial P_i(\partial V/\partial\psi)}{\partial t}\bigg|_\psi + \frac{\partial}{\partial\psi}\left[P_i \frac{\partial V}{\partial t}\bigg|_\psi + \left\langle\left(Q_i + \frac{5}{2}\frac{\Gamma_R}{Z} T_i\right)\cdot\nabla V\right\rangle\right] = S_i \frac{\partial V}{\partial\psi} \tag{3}$$

$$\frac{\partial f\langle 1/r^2\rangle(\partial V/\partial\psi)}{\partial t}\bigg|_\psi + \frac{\partial}{\partial\psi} c\langle E\cdot B\rangle \frac{\partial V}{\partial\psi} = 0 \tag{4}$$

$$\frac{1}{\partial V/\partial\psi}\frac{\partial G(V)/(\partial V/\partial\psi)}{\partial\psi} + \frac{4\pi\partial(P_e + P_i)}{\partial\psi} + f\frac{\partial f}{\partial\psi}\left\langle\frac{1}{r^2}\right\rangle = 0 \tag{5}$$

where ψ is the poloidal flux, V the volume contained within a flux surface (so $\partial V/\partial\psi$ is related to the toroidal current density), n, the electron density, P_e and P_i the electron and ion pressures, r the distance from the major axis, and f the poloidal current. Angle brackets denote flux surface averages, i.e., $\langle a\rangle = \int a d^2S/|\nabla V|$. G is Grad's shape function [2], $G(V) = \langle(\nabla V/r)^2\rangle$. The S's are source functions, and the Q's are heat fluxes. The sources S_e and S_i here include all the energy sources, not just the collisional part as in Ref. [1], hence the slightly different form of Eqs. (2) and (3).

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The particle flux nU may be calculated from the toroidal component of an appropriate Ohm's law.

$$E_\varphi + \left(\frac{U}{c} \times B\right)_\varphi = R_\varphi$$

Γ_R is that part of the flux due to R_φ. $\langle \Gamma_R \cdot \nabla V \rangle$, $\langle Q_e \cdot \nabla V \rangle$, $\langle Q_i \cdot \nabla V \rangle$, and $\langle E \cdot B \rangle$ are linearly related to ψ derivatives of n, P_e, P_i, and f by transport coefficients [3].

Equations (1)–(5) determine the five quantities n, P_e, P_i, f, and $\partial V/\partial\psi$. The first four equations are in conservation form, in the sense that they may be integrated over ψ to find the rate of change of the number of particles, the energy, or the toroidal flux contained between two ψ surfaces in terms of surface terms and volume integrals of sources. We wish that our difference equations preserve this property so that only physical processes create or destroy particles, energy, and flux.

This consideration suggests that the products of dependent variables, $n \times \partial V/\partial\psi$, $P_e \times \partial V/\partial\psi$, etc., appearing under the time derivative be preserved in the difference formulation, rather than reducing them by taking the derivative of each variable separately. This implies a nonlinear set of difference equations to represent Eqs. (1)–(4).

We require an implicit time differencing to allow reasonable timesteps to be taken with numerical stability [4]. The equilibrium constraint, Eq. (5), vanishes at the backward time because the system was then in equilibrium, so only concerns forward values. It can be regarded as a nonlinear constraint on the variables at the forward time.

These aspects of the problem are contained in the much simpler model

$$\frac{\partial ab}{\partial t} + \frac{\partial^2 a}{\partial x^2} = 0 \tag{6}$$

$$C\left(a, b, \frac{\partial a}{\partial x}, \frac{\partial b}{\partial x}\right) = 0 \tag{7}$$

which will be used to illustrate the solution technique used by *G2M*.

Using super F and B to represent forward and backward time levels, these become

$$\frac{a^F b^F - a^B b^B}{\Delta t} + \frac{\partial^2 a^F}{\partial x^2} = 0 \tag{8}$$

$$C\left(a^F, b^F, \frac{\partial a^F}{\partial x}, \frac{\partial b^F}{\partial x}\right) = 0 \tag{9}$$

These equations are still in conservative form, but are nonlinear. To solve them, we recall a technique described by Roberts [5] for handling a similar problem. He assumed adiabaticity, linearized the constraint equations, and iterated (a Newton's method in function space). We also will linearize and iterate, but on the full set of equations, including transport. As applied to the simple model, we therefore:

(A) Replace (a^F, b^F) by $(a, b) + (\delta a, \delta b)$ and linearize in $(\delta a, \delta b)$. This gives

$$\frac{ab + a\,\delta b + b\,\delta a - a^B b^B}{\Delta t} + \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2\,\delta a}{\partial x^2} = 0 \qquad (10)$$

$$C(a, b) + \frac{\partial C}{\partial a}\delta a + \frac{\partial C}{\partial b}\delta b + \frac{\partial C}{\partial(\partial a/\partial x)}\delta\frac{\partial a}{\partial x} + \frac{\partial C}{\partial(\partial b/\partial x)}\delta\frac{\partial b}{\partial x} = 0 \qquad (11)$$

(B) Initialize (a, b) to (a^B, b^B).

(C) Solve the linear simultaneous equations (10) and (11) for $(\delta a, \delta b)$, noting that when $\partial/\partial x$ is replaced by its standard finite difference equivalent $\Delta/\Delta x$, the equations are block tridiagonal and can be rapidly and readily solved.

(D) Update the variables and if $(\delta a, \delta b)$ are not small enough, repeat the process from step C. This technique converges quadratically, meaning that the relative error is squared each iteration, so only a few iterations are required to find (a^F, b^F) to machine accuracy.

Notice that the linearized equations do not formally conserve particles, energy, or flux, because of the neglect of the quadratic terms. Conservation is, therefore, only achieved to the extent that these terms are negligible, which is why we iterate the solution to convergence at each timestep.

The *G2M* code uses this technique to solve the larger set of Eqs. (1)–(5) simultaneously in conservation form. If the chenges of the parameters $G(V)$ and $\langle 1/r^2\rangle(V)$ are omitted, the convergence is indeed quadratic, as seen in Fig. 1. In practice these functions depend only weakly on volume for regular nested flux surfaces, so the code allows them to vary without linearizing them, thus reducing the convergence slightly, to about a factor of typically 100 per iteration as seen in Fig. 2.

The code tallies sources and boundary losses in time for particles, energy, and magnetic flux, and displays running sums of the volume integrals of these quantities. The balance from the beginning to the end of a problem, on the CDC 7600, is generally to within a part in 10^{12}.

Repeated linearization of nonlinear time-dependent problems has been recognized as a useful technique for some time. Beam and Warming [6] discuss its use in the numerical solution of nonlinear hyperbolic systems, and Briley and McDonald [7] apply it to the Navier-Stokes equation. Both sets of authors expand about the forward time level, linearize, and thereby define implicit equations which can be solved by standard techniques. The numerical stability properties thus gained are of great importance. Finan and Killeen [8] have applied the technique to the 3*D* set of resistive *MHD* equations and, as here, have iterated the linearization to convergence.

Turkel has observed [9] that such iteration does not increase the formal accuracy of the solution, since an exact solution of equations good to $O(\Delta t)$ is itself accurate only to the same order, but that such iteration may have other desirable properties such as the exact conservation obtained here or the ability to recover steady-state solutions for a very large timestep. This latter property does not strictly obtain here since the transport coefficients are evaluated at the old ("*B*") time but is a useful feture of a

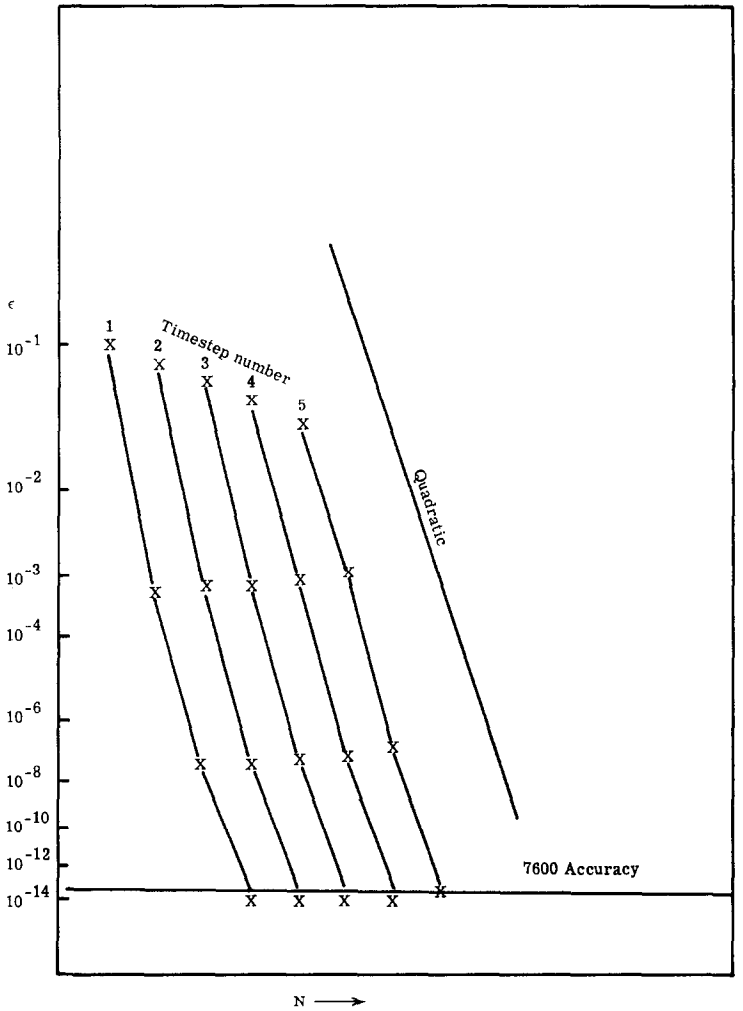

FIG. 1. Maximum relative change ϵ as a function of iteration number N, for the quantities of Eqs. (1–5) holding G and $\langle 1/r^2 \rangle$ fixed, in *G2M*. Five different times are shown. Machine accuracy is achieved on the fourth iteration. The theoretical quadratic convergence curve, $\epsilon = \epsilon_0^{2^N}$, is also shown.

code developed by one of us (*HHK*) describing the Elmo Bumpy Torus and another code (by *RNB*) describing oxygen diffusion in tokamaks. It is true here that the elliptic constraint of pressure balance is always enforced on the diffusing variables, so equilibrium is maintained.

We have described the application of the linearization technique to a set of equations describing tokamak transport. This technique has allowed the equations to be solved using implicit (time) differencing while retaining their conservation properties and subject to the elliptic constraint.

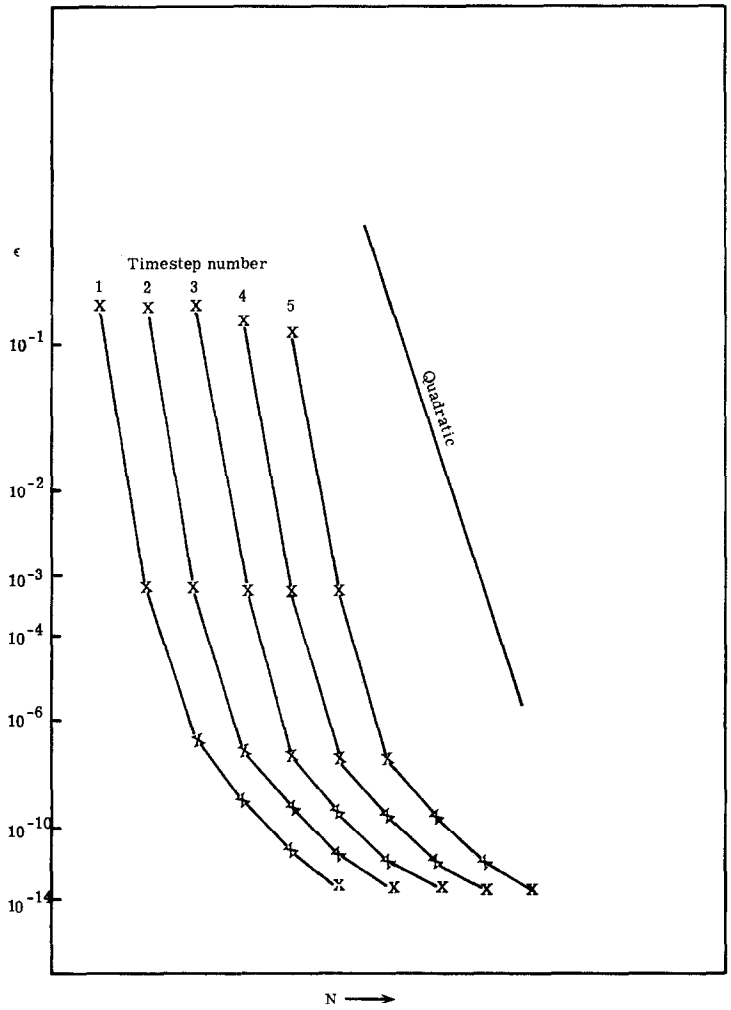

FIG. 2. As Fig. 1, except the geometrical factors are allowed to vary, spoiling quadratic convergence. Machine accuracy is achieved on the sixth iteration.

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